# Research Statement

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My research is in equivariant homotopy theory. One of my core aims is to extend Blumberg and Hill's $\mathbb{N}_\infty$-operad theory [BH15], which is an extension of non-equivariant $\mathbb{E}_\infty$-operad theory that incorporates norms in a highly structured way. The importance of this theory is that it provides an overarching framework for norm maps, which, ever since the Hill-Hopkins-Ravenel solution to the Kervaire Invariant One problem [HHR15], has proven to be essential to the study of equivariant spectra. However, an analogous extension of $\mathbb{E}_k$-operads is still unclear, and there are open questions about how such an extension would behave. Answering this is the central goal of my research. Ultimately, the hope is that such an extension will lead to a better understanding of multiplicative objects that appear in equivariant stable homotopy theory, obstructions to such objects, and in practice, help further computations.

## 1 Background

In Algebraic Topology, one considers algebraic objects in homotopy categories. Since homotopy categories are categories where weak equivalences are formally inverted, one immediately finds that the maps that underlie the algebraic object don't necessarily satisfy the same defining relations, if any. This lack of structure makes arguing about the objects in homotopy from the underlying point set model difficult – so we *add it*. This sacrifices simplicity for more control. Fortunately, homotopy theorists have developed a wide range of tools to deal with this added complexity. One of the most important is the notion of an operad.

The central idea of an operad is not to choose single representations of homotopy classes but instead work with a range of possible choices. Specifically, a topological operad $P$ consists of a collections of spaces $\{P(n)\}_{n\in\mathbb{N}}$ in **Top**, along with maps

$$\gamma : P(n) \times P(k_1) \times \cdots \times P(k_n) \to P(k_1 + \cdots + k_n)$$

called composition maps. These objects and maps must then satisfy a bunch of conditions that emulate the expected properties of the collections $\{\mathrm{Hom}(X^n, X)\}_{n\in\mathbb{N}}$ and composition in each component. The intuition is that these objects $P(n)$ parameterise a range of possible $n$-ary operators on objects. Specific objects $X$ with an algebraic structure determined by this operad $P$ – called $P$-algebras – are then given by maps $P(n) \to \mathrm{Hom}(X^n, X)$. These must be compatible with the expected associativity, unitality and commutativity relations. Moreover, the spaces $P(n)$ must have an action by $\Sigma_n$, which is well-behaved with the operadic structure. This action corresponds with permuting the inputs of $\mathrm{Hom}(X^n, X)$.

The power of operads is that not only does $P(n)$ encode possible choices of $n$-array operators, but as these are spaces, they also encode higher homotopic information. i.e., paths parameterise homotopies between choices, 2-cells parameterise homotopies between homotopies and so on. This information is precisely the added structure we want. We can leverage this to build better point-set models that underlie homotopy categories.

An essential class of operads are the $\mathbb{E}_k$-operads. There are a few different ways to think of these. The most straightforward method is to think about the standard model for them: the little cubes operad of Boardman and Vogt. We will write $\square^k = [0,1]^k$ and say that a little $n$-cube is a rectilinear embedding

$$\coprod_n \square^k \to \square^k.$$

The operad is then $\mathscr{C}_k(n) := \mathrm{Emb}(\coprod_n \square^k, \square^k)$ and composition is then given by function composition. The main example of a $\mathscr{C}_k$-algebra are $k$-fold loop spaces $X = \Omega^k Y$

The action of $\Sigma_n$ on $\mathscr{C}_k(n)$ permutes the order of the little cubes. Observe that $\Sigma_n$ acts freely, and depending on $k$, we might be able to find homotopies, and higher homotopies, between little cubes in the same orbit. If $k < \infty$, we will eventually get stuck in finding higher homotopies. If $k = \infty$, these spaces are contractible, so we can always find such higher homotopies. This behaviour gives us one interpretation of $\mathbb{E}_k$-operads: these are operads that encode permutations of inputs on multiplications maps, as well as homotopies and higher homotopies of these – at least up to a certain point depending on $k$. When $k = \infty$, we record all possible higher homotopical data. i.e., we have the homotopy-coherent case. Hence, the $\mathbb{E}_k$-operads encode highly structured data for a multiplication that lies between just associativity to something commutative.

Another interpretation of $\mathbb{E}_k$-operads is that these are operads that encode objects with $k$ different multiplications that interchange. A good illustration of this behaviour is the $k$-fold loop spaces $\Omega^k X$. For the 2-fold case we have

$$\Omega^2 X \cong \mathrm{Hom}(\square^2, X) \cong \mathrm{Hom}(\square^1_{\mathrm{v}}, \mathrm{Hom}(\square^1_{\mathrm{h}}, X)) \cong \mathrm{Hom}(\square^1_{\mathrm{h}}, \mathrm{Hom}(\square^1_{\mathrm{v}}, X)).$$

Here $v$ and $h$ denote vertical and horizontal, and each direction corresponds to a different composition $\circ_v$ and $\circ_h$ of $\Omega^2 X$. The two different compositions interchange in that

$$(f \circ_v g) \circ_h (h \circ_v k) = (f \circ_h h) \circ_v (g \circ_h k).$$

Given two operads $P$ and $Q$, the Boardman-Vogt tensor product $P \otimes Q$ is the operad that universally encodes the two algebraic structures of $P$ and $Q$ and the interchange of these two. The operad $\mathbb{E}_{\mathbb{1}} = \mathscr{C}_1$ encodes associativity up to homotopy, and so under this interpretation, we expect that $\mathbb{E}_k$ to be tensor powers of $\mathbb{E}_1$. A classic result by Dunn [Dun88] and extended by Brinkmeier [Bri00] connects these two interpretations. They showed that for the little cube operads, there is a weak equivalence of operads

$$\mathscr{C}_n \otimes \mathscr{C}_m \simeq \mathscr{C}_{n+m}.$$

Now, $\mathbb{E}_k$-operads encode multiplication maps in a highly structured way. However, equivariantly there is more that we want to encode. The most important of which is the structure of norm maps.

Greenlees and May first introduced these in [GM97] and played a central role in the Hill-Hopkins-Ravenel solution [HHR15] to the Kervaire Invariant One problem. We will find it preferable to think of norm maps on a $G$-object as "twisted" multiplication maps of the form $X^{|G/H|} \to X$. Here $G$ not only acts on each copy of $X$ but also permutes the indices of the products. More generally, for each finite $G$-set $T$, we can talk about norm maps as maps of the form $\times_T X \to X$ where $G$ acts on the indexing as well as on each copy of $X$.

Blumberg and Hill [BH15] extended $\mathbb{E}_\infty$-operad theory to include the extra data of norm maps in a homotopical meaningful way, which they dubbed $\mathbb{N}_\infty$-operads. The key idea is that the norm maps a $G$-operad $\mathscr{O}$ encodes is precisely given by its fixed points $\mathscr{O}(n)^\Gamma$ where $\Gamma$ is what is a called a graph subgroup of $G \times \Sigma_n$. $\mathbb{N}_\infty$-operads are then operads where the fixed points are either empty or contractible. The combinatorial data of which norms exist and don't exist for an $\mathbb{N}_\infty$-operad $\mathscr{O}$ then gets encapsulated in the notion of its *indexing system* $\mathscr{I}(\mathscr{O})$, the category of which forms a lattice. It turns out that the homotopy category of $\mathbb{N}_\infty$-operads is equivalent to the lattice of indexing systems (See [BH15], [Rub21], [GW18], [BP21]). That is, $\mathbb{N}_\infty$-operads are completely determined by what norm maps they encode.

## 2 Current Work

Blumberg and Hill's work extends the $\mathbb{E}_\infty$ case, but what about $\mathbb{E}_k$ in general? Answering this is the central goal of my research.

**Question 1.** What is a good model for "$\mathbb{N}_k$-operads"? What properties do they have?

In this section, I will talk about two threads of inquiry I have completed concerning this question. In the next section, I will discuss further work I plan to undertake to answer this question.

### 2.1 Equivariant Additivity

In the first section, I discussed Dunn additivity as a central component for $\mathbb{E}_k$-operads. In the equivariant case, Blumberg-Hill conjecture [BH15] that the tensor of $\mathbb{N}_\infty$-operads should correspond to the lattice join of their indexing systems. We hope that a similar property holds in the $\mathbb{N}_k$-operad case. However, cubes are ill-behaved with respect to group actions, and the natural replacement for such a statement is to use the equivariant little disk operads $\mathscr{D}_V$ instead. These operads are defined similarly to the little cubes operad $\mathscr{C}_n$ except the embedding shape is the unit disk of an orthogonal representation $V$ of $G$, and $G$ acts on the operad via conjugation. Moreover, the operads $\mathscr{D}_V$ for infinite-dimensional representations are $\mathbb{N}_\infty$-operads, and the finite-dimensional versions seem to be reasonable candidates for an $\mathbb{N}_k$-operad.

In [Szc24], I have obtained a version of this classic result for equivariant little disk operads.

**Result 1.** For orthogonal $G$-representations $V$ and $W$, there is a weak equivalence of equivariant operads

$$\mathscr{D}_V \otimes \mathscr{D}_W \simeq \mathscr{D}_{V \oplus W}$$

Here, the $G$-representations $V$ and $W$ are allowed to be $G$-universes, so this also verifies the Blumberg-Hill conjecture for a portion of the lattice of indexing systems. In fact, we prove a slightly more general statement than stated in result 1 and prove an additivity theorem for "equivariant framed little disk operads".

It is worth highlighting here that this generalization isn't trivial. The main issue is that the Boardman-Vogt tensor isn't homotopical, and the original non-equivariant proofs heavily rely on the geometry of rectangles, which we can't use here. So even if we take this statement non-equivariantly, as far as I'm aware, there is no proof in the literature for the additivity of the little disk operads.

My proof of result 1 in [Szc24] introduces a couple of new key ideas:

1. The proof in the literature of the weak equivalence between little disk $\mathscr{D}_n$ and little cube operads $\mathscr{C}_n$ involves a zigzag through abstractly defined combinatorial operads. We prove a "framed equivariant" generalization of this that is entirely geometric in its arguments.

2. We delve further into determining when operadic maps from a tensor of operads is injective. The hard part of proving Dunn additivity is showing that the map $\mathscr{C}_n \otimes \mathscr{C}_m \to \mathscr{C}_{n+m}$ is injective, and this relies on the geometry of rectangles. The analogous map for the little disks *isn't injective*. Instead, we have to identify a weakly equivalent subcollection we call the additive core $\mathscr{K} \subseteq \mathscr{D}_V \otimes \mathscr{D}_W$ in which the induced map is injective.

One immediate application of result 1 is that it allows us to automatically impose extra structure for $\mathbb{N}_\infty$-operads modelled by little disk operads. For instance, given some $G$-universe $U$, since $U = U \oplus U$ we get that $\mathscr{D}_U = \mathscr{D}_U \otimes \mathscr{D}_U$. Hence, by the defining property of the tensor, any $\mathscr{D}_U$-algebra $X$ is also a $\mathscr{D}_U$-algebra *in $\mathscr{D}_U$-algebras*.

## 2.2 Realizing $\mathbb{N}$-Operads as Suboperads of Coinduced Operads.

As we stated in the introduction, each indexing system $\mathscr{I}$ has a realization as the combinatorial norm data of an $\mathbb{N}_\infty$-operad $\mathscr{O}$. This is more complicated than one might naively think. For instance, the little disk operads $\mathscr{D}_V$ only account for a portion of the indexing system lattice. This realization problem was originally conjectured by Blumberg-Hill in [BH15] and has since been proven in a number of different ways by Gutierrez-White [GW18], Rubin [Rub21], and Bonventre-Pereira [BP21]. These approaches all use something inherent to $\mathbb{E}_\infty$-operads and are ill-suited for possible $\mathbb{E}_k$ generalizations. In upcoming work [Szc], which extends initial results from my dissertation [Szc23], we add a fourth method that can realize indexing systems as $\mathbb{N}_\infty$-operads, but more importantly, can also realize these as indexing systems for what might be called "$\mathbb{N}_k$-operads."

The main idea is that for any $G$-operad $\mathscr{O}$, the coinduced operad $N_e^G \mathscr{O}$ encodes all possible norm maps, and so for an indexing system $\mathscr{I}$, we want to find a way to construct a suboperad $N^{\mathscr{I}} \mathscr{O} \subseteq N_e^G \mathscr{O}$ that only encodes norm maps that correspond to the indexing system $\mathscr{I}$. In [Szc], we achieve this construction for a variety of non-equivariant operads which include the little cube operads $\mathscr{C}_n$, little disk operads $\mathscr{D}_n$, Steiner operads $\mathscr{K}_n$, and linear isometry operad $\mathscr{L}$.

**Result 2.** Let $P$ be any of the operads previously mentioned. Then for any indexing system $\mathscr{I}$, there exists a suboperad $N^{\mathscr{I}} P \subseteq N_e^G P$ such that $(N^{\mathscr{I}} P)^\Gamma = (N_i^G P)^\Gamma$ for any graph subgroup $\Gamma$ determined by $\mathscr{I}$ and $(N^{\mathscr{I}} P)^\Gamma = \emptyset$ otherwise.

A consequence of this is that we have a realization for any indexing system $\mathscr{I}$ by the $\mathbb{N}_\infty$-operad $N^{\mathscr{I}} \mathscr{C}_\infty$, and moreover, we have a filtration

$$N^{\mathscr{I}} \mathscr{C}_1 \subseteq N^{\mathscr{I}} \mathscr{C}_2 \subseteq \dots N^{\mathscr{I}} \mathscr{C}_3 \subseteq \cdots \subseteq N^{\mathscr{I}} \mathscr{C}_\infty. \tag{1}$$

Where each intermediate operad $N^{\mathscr{I}} \mathscr{C}_k$ encodes exactly the same norm data as $N^{\mathscr{I}} \mathscr{C}_\infty$, just with less homotopy coherence data. These operads could then be considered a prototype for $\mathbb{N}_k$-operads.

## 3 Future Work

### 3.1 $\mathbb{N}_k$-operads

The suboperads $N^{\mathscr{I}} \mathscr{C}_k$ are constructed in a fairly ad hoc manner, and it is not yet clear if they possess all the properties that a good notion of "$\mathbb{N}_k$-operads" should have. The first question I have is if we can make this construction more universal.

**Question 2.** Can the $N^{\mathscr{I}}$ construction be extended to all operads in a functorial manner? That is, is there a kind of "incomplete coinduction" functor for operads?

Although not every $\mathbb{N}_\infty$-operad can be modelled by the little disk operads, when they can, we have alternate natural filtrations. For instance, if $V$ is any finite $G$-representation which contains a trivial representation, then we have a filtration of the $\mathbb{N}_\infty$-operad $\mathscr{D}_{V^\infty}$ given by

$$\mathscr{D}_V \subseteq \mathscr{D}_{V^2} \subseteq \mathscr{D}_{V^3} \subseteq \cdots \subseteq \mathscr{D}_{V^\infty} \tag{2}$$

where each $\mathcal{D}_{V^k}$ realizes the same norm maps. A natural question is then:

**Question 3.** How does eq. (1) compare to eq. (2) for those indexing systems realizable by little disk operads?

Also, non-equivariantly, an $\mathbb{E}_1$-operad is meant to encode homotopy associative monoids. The biggest reason I hesitate to call the operads $\mathbb{N}_k$-operads is because the following question still needs to be answered.

**Question 4.** Does $N^{\mathscr{I}}\mathscr{C}_1$ have the same relationship to the associative operad **Assoc** as $N^{\mathscr{I}}\mathscr{C}_\infty$ does to the commutative operad **Comm**?

Besides our current prototypes, there are other possibilities we can try to construct models of $\mathbb{N}_k$-operads. One idea is to use Barwick's operator categories [Bar18]. Barwick's theory shows how to construct different kinds of Complete Segal operads and Lurie's $\infty$-operads starting from an "operator category". These constructions give us a weak notion of operad instead of the strict kind I've used up to this point. The category of finite ordered sets **O** is the category that controls the range of possible multiplications of non-symmetric operads. Here, the $\mathbb{E}_1$-operads are given by cofibrant replacement of the unit operad over **O**, and the $\mathbb{E}_k$-operads are given as cofibrant replacement of the unit operads over the wreath iterates $\mathbf{O}^{(n)} := \mathbf{O} \wr \mathbf{O} \wr \cdots \wr \mathbf{O}$. Since we know combinatorially what multiplications to expect, it seems possible to use (some extension of) Barwick's theory to construct a notion of $\mathbb{N}_k$-($\infty$-)operads.

**Question 5.** Can we construct a model for (the homotopy category of) $\mathbb{N}_k$-operads by using Barwick's operator categories and picking the correct operator categories?

### 3.2 Equivariant Additivity

While I have proven additivity for the little disks, the question arises: what other operads have an additivity property? Initial work seem to suggest that we do have an additivity property for the $N^{\mathscr{I}}\mathscr{C}_\infty$ operads as follows:

**Conjecture 1.** Given indexing systems $\mathscr{I}$ and $\mathscr{I}'$, there is a weak equivalence

$$N^{\mathscr{I}}\mathscr{C}_\infty \otimes N^{\mathscr{I}'}\mathscr{C}_\infty \simeq N^{\mathscr{I}\vee\mathscr{I}'}\mathscr{C}_\infty.$$

Further verifications still need to be done, but if it holds, this would essentially verify the last remaining conjecture of Blumberg-Hill in [BH15]. I say essentially here as the actual conjecture is phrased in terms of cofibrant operads. However, even non-equivariantly, it isn't clear[1] whether cofibrant $\mathbb{E}_k$-operads are additive.

[1] It isn't well known, but the paper that proves this [FV15] has a mistake in it, and so the status of this result is unclear. Either way, it is almost universally expected that such a result should hold in some form.

An essential part of my proof of additivity for little disks is the notion of "separateness," which allows us to restrict the size of little disks. Intuitively, this implies that the homotopy of little disks depends only on the relative configurations of the centres of the disks. However, this idea is already captured in the non-equivariant case by the Fulton-Macpherson Operad $\mathscr{F}_m$, which is weakly equivalent to little disks. This leads me to conjecture the following

**Conjecture 2.** There is a weak equivalence of operads

$$\mathscr{F}_m \otimes \mathscr{F}_n \simeq \mathscr{F}_{m+n}$$

which can also be extended equivariantly.

Moreover, $\mathscr{F}_m$ are almost cofibrant. So this would be an interesting connection between the two approaches to additivity in the classic strict models – the combinatorial one via universal cofibrant models and the geometric one by little cubes.

Equivariance can also feature more strongly in additivity. In particular, since the Boardman-Vogt Tensor is symmetric monoidal, we can define a corresponding tensor induction

$$\mathrm{ind}_H^G : \mathrm{Op}(H) \to \mathrm{Op}(G).$$

This allows us to construct $G$-operads, and understanding the homotopical properties of this functor would be interesting. In connection with additivity, we would expect the following holds

**Conjecture 3.** For a $H$-representation $V$, we have a weak equivalence of $G$-operads

$$\mathrm{ind}_H^G(\mathscr{D}_V) \simeq \mathscr{D}_{\mathrm{ind}_H^G V}.$$

### 3.3 Realizing Compatible Pairs of Indexing Systems

A Tambara functor is an algebraic gadget that appears as the 0-th homotopy group of genuine equivariant ring spectra. These have a notion of both additive transfer maps and multiplicative norm maps with certain compatibility conditions. Blumberg-Hill [BH21] define and investigate the notion of bi-incomplete Tambara functors where the transfer and norm maps are determined by a compatible pair of indexing systems $(\mathscr{A}, \mathscr{M})$. A natural question, which generalizes the realizability theorem for $\mathbb{N}_\infty$-operads is then the following.

**Question 6.** Given a compatible pair of indexing systems $(\mathscr{A}, \mathscr{M})$, does there exist an operad pair of $\mathbb{N}_\infty$-operads $(\mathscr{O}^{\mathscr{A}}, \mathscr{O}^{\mathscr{M}})$ that realizes this compatible pair of indexing systems?

I am currently investigating this question as part of the AMS' Mathematical Research Communities (MRC) on Homotopical Combinatorics. Initial work of our group suggests that my construction above has distinct advantages for this question as compared to the other methods. In particular, the notion of an operad pair requires a pairing map of the form

$$\lambda : \mathcal{O}^{\mathcal{M}}(n) \times \mathcal{O}^{\mathcal{A}}(j_1) \times \cdots \times \mathcal{O}^{\mathcal{A}}(j_n) \to \mathcal{O}^{\mathcal{A}}(j_1 \ldots j_n)$$

that is compatible with the operad structures. This is an incredibly hard map to construct ex nihilo, and the more abstract realizations of $\mathbb{N}_\infty$-operads offer little help to build one. However, non-equivariantly, the linear isometries and Steiner operads form the canonical pairing $(\mathcal{L}, \mathcal{K}_\infty)$ and this behaves well with coinduction. So we automatically get a pairing of the coinduced operads $(N^G\mathcal{L}, N^G\mathcal{K}_\infty)$. Proving question 6 then amounts to showing that the pairing map $\lambda$ restricts to a well-defined pairing map of the pair $(N^{\mathcal{M}}\mathcal{L}, N^{\mathcal{A}}\mathcal{K}_\infty)$. While we don't believe this exact example works, we have some promising variations that we are still in the middle of verifying.

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