Research Statement

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My research is in equivariant homotopy theory. One of my core aims is to extend Blumberg and Hill's $\mathbb{N}_\infty$-operad theory [BH15], which is an extension of non-equivariant $\mathbb{E}_\infty$-operad theory that incorporates norms in a highly structured way. The importance of this theory is that it provides an overarching framework for norm maps, which, ever since the Hill-Hopkins-Ravenel solution to the Kervaire Invariant One problem [HHR15], has proven to be essential to the study of equivariant spectra. However, an analogous extension of $\mathbb{E}_k$-operads is still unclear, and there are open questions about how such an extension would behave. Answering this is the central goal of my research. Ultimately, the hope is that such an extension will lead to a better understanding of multiplicative objects that appear in equivariant stable homotopy theory, obstructions to such objects, and in practice, help further computations.

1 Background

In Algebraic Topology, one considers algebraic objects in homotopy categories. Since homotopy categories are categories where weak equivalences are formally inverted, one immediately finds that the maps that underlie the algebraic object don't necessarily satisfy the same defining relations, if any. This lack of structure makes arguing about the objects in homotopy from the underlying point set model difficult – so we *add it*. This sacrifices simplicity for more control. Fortunately, homotopy theorists have developed a wide range of tools to deal with this added complexity. One of the most important is the notion of an operad.

The central idea of an operad is not to choose single representations of homotopy classes but instead work with a range of possible choices. Specifically, a topological operad P consists of a collections of spaces $\{P(n)\}_{n\in\mathbb{N}}$ in **Top**, along with maps

$$\gamma : P(n) \times P(k_1) \times \cdots \times P(k_n) \to P(k_1 + \cdots + k_n)$$

called composition maps. These objects and maps must then satisfy a bunch of conditions that emulate the expected properties of the collections $\{\mathrm{Hom}(X^n, X)\}_{n\in\mathbb{N}}$ and composition in each component. The intuition is that these objects $P(n)$ parameterise a range of possible n-ary operators on objects. Specific objects X with an algebraic structure determined by this operad P – called P-algebras – are then given by maps $P(n) \to \mathrm{Hom}(X^n, X)$. These must be compatible with the expected associativity, unitality and commutativity relations. Moreover, the spaces $P(n)$ must have an action by Σ_n, which is well-behaved with the operadic structure. This action corresponds with permuting the inputs of $\mathrm{Hom}(X^n, X)$.

The power of operads is that not only does $P(n)$ encode possible choices of n-array operators, but as these are spaces, they also encode higher homotopic information. i.e., paths parameterise homotopies between choices, 2-cells parameterise homotopies between homotopies and so on. This information is precisely the added structure we want. We can leverage this to build better point-set models that underlie homotopy categories.

An essential class of operads are the $\mathbb{E}_k$-operads. There are a few different ways to think of these. The most straightforward method is to think about the standard model for them: the little cubes operad of Boardman and Vogt. We will write $\square^k = [0,1]^k$ and say that a little n-cube is a rectilinear embedding

$$\coprod_n \square^k \to \square^k.$$

The operad is then $\mathscr{C}_k(n) := \mathrm{Emb}(\coprod_n \square^k, \square^k)$ and composition is then given by function composition. The main example of a $\mathscr{C}_k$-algebra are k-fold loop spaces $X = \Omega^k Y$

The action of Σ_n on $\mathscr{C}_k(n)$ permutes the order of the little cubes. Observe that Σ_n acts freely, and depending on k, we might be able to find homotopies, and higher homotopies, between little cubes in the same orbit. If $k < \infty$, we will eventually get stuck in finding higher homotopies. If $k = \infty$, these spaces are contractible, so we can always find such higher homotopies. This behaviour gives us one interpretation of $\mathbb{E}_k$-operads: these are operads that encode permutations of inputs on multiplications maps, as well as homotopies and higher homotopies of these – at least up to a certain point depending on k. When $k = \infty$, we record all possible higher homotopical data. i.e., we have the homotopy-coherent case. Hence, the $\mathbb{E}_k$-operads encode highly structured data for a multiplication that lies between just associativity to something commutative.

Another interpretation of $\mathbb{E}_k$-operads is that these are operads that encode objects with k different multiplications that interchange. A good illustration of this behaviour is the k-fold loop spaces $\Omega^k X$. For the 2-fold case we have

$$\Omega^2 X \cong \mathrm{Hom}(\square^2, X) \cong \mathrm{Hom}(\square^1_{\mathrm{v}}, \mathrm{Hom}(\square^1_{\mathrm{h}}, X)) \cong \mathrm{Hom}(\square^1_{\mathrm{h}}, \mathrm{Hom}(\square^1_{\mathrm{v}}, X)).$$

Here v and h denote vertical and horizontal, and each direction corresponds to a different composition $\circ_v$ and $\circ_h$ of $\Omega^2 X$. The two different compositions interchange in that

$$(f \circ_v g) \circ_h (h \circ_v k) = (f \circ_h h) \circ_v (g \circ_h k).$$

Given two operads P and Q, the Boardman-Vogt tensor product $P \otimes Q$ is the operad that universally encodes the two algebraic structures of P and Q and the interchange of these two. The operad $\mathbb{E}_{\mathbb{1}} = \mathscr{C}_1$ encodes associativity up to homotopy, and so under this interpretation, we expect that $\mathbb{E}_k$ to be tensor powers of $\mathbb{E}_1$. A classic result by Dunn [Dun88] and extended by Brinkmeier [Bri00] connects these two interpretations. They showed that for the little cube operads, there is a weak equivalence of operads

$$\mathscr{C}_n \otimes \mathscr{C}_m \simeq \mathscr{C}_{n+m}.$$

Now, $\mathbb{E}_k$-operads encode multiplication maps in a highly structured way. However, equivariantly there is more that we want to encode. The most important of which is the structure of norm maps.

Greenlees and May first introduced these in [GM97] and played a central role in the Hill-Hopkins-Ravenel solution [HHR15] to the Kervaire Invariant One problem. We will find it preferable to think of norm maps on a G-object as "twisted" multiplication maps of the form $X^{|G/H|} \to X$. Here G not only acts on each copy of X but also permutes the indices of the products. More generally, for each finite G-set T, we can talk about norm maps as maps of the form $\times_T X \to X$ where G acts on the indexing as well as on each copy of X.

Blumberg and Hill [BH15] extended $\mathbb{E}_\infty$-operad theory to include the extra data of norm maps in a homotopical meaningful way, which they dubbed $\mathbb{N}_\infty$-operads. Different $\mathbb{N}_\infty$-operads are allowed to encode different norm maps. This combinatorial data gets encapsulated in the notion of an *indexing system*. Indexing systems are certain kinds of categorical coefficient systems in finite equivariant sets. It turns out that the homotopy category of $\mathbb{N}_\infty$-operads is equivalent to the category of indexing systems (See [BH15], [Rub21], [GW18], [BP21]). That is, these operads are completely determined by what norm maps they encode.

2 Current Work

Blumberg and Hill's work extends the $\mathbb{E}_\infty$ case, but what about $\mathbb{E}_k$ in general? Answering this is the central goal of my research.

Question 1. What is a good model for "$\mathbb{N}_k$-operads"? What properties do they have?

In this section, I will talk about two threads of inquiry I have completed concerning this question. In the next, I will discuss further work I plan to undertake to answer this question.

2.1 Equivariant Additivity

In the first section, I discussed Dunn additivity as a central component for $\mathbb{E}_k$-operads. We hope that a similar property holds in the $\mathbb{N}_k$-operad case. However, cubes are ill-behaved with respect to group actions, and the natural replacement for such a statement is to use the equivariant little disk operads $\mathscr{D}_V$ instead. These operads are defined similarly to the little cubes operad $\mathscr{C}_n$ except the embedding shape is the unit disk of an orthogonal representation V of G, and G acts on the operad via conjugation. Moreover, the operads $\mathscr{D}_V$ for infinite-dimensional representations are

$\mathbb{N}_\infty$-operads, and the finite-dimensional versions seem to be reasonable candidates for an $\mathbb{N}_k$-operad.

In [Szca], I have obtained a version of this classic result for equivariant little disk operads.

Result 1. For inner product spaces V and W which are orthogonal G-representations, there is a weak equivalence of equivariant operads

$$\mathscr{D}_V \otimes \mathscr{D}_W \simeq \mathscr{D}_{V\oplus W}$$

I end up showing a bit more than just this. It is worth highlighting here that this statement isn't trivial. The main issue is that the Boardman-Vogt tensor isn't homotopical. So even if we take this statement non-equivariantly, as far as I'm aware, there is no proof in the literature for the additivity of the little disk operads.

My proof relies on a couple of key observations:

1. The definition of equivariant little disks $\mathscr{D}_V$ works just as well where V is a general normed vector space and G acts via isometries. We can then view little cube operads as special cases of little disk operads.

2. For disks d^1, d^2, if we write the radii as r_1, r_2 and the length between their centres as D, then the condition that they don't overlap can be phrased as $D/(r_1 + r_2) \geq 1$. Suppose we restrict ourselves to little disks where this "separation" quantity is pairwise greater than some constant $k \geq 1$. In that case, this forms a sub-operad $\mathscr{D}_V^k$, which can be shown to be weakly equivalent to the original.

Using the first observation, I extended the classical proofs to this generality to get:

Result 2. There is a weak equivalence of equivariant operads

$$\mathscr{D}_V \otimes \mathscr{D}_W \simeq \mathscr{D}_{V \overset{\infty}{\oplus} W}$$

where $V \overset{\infty}{\oplus} W$ is $V \oplus W$ with the a product sup-norm[1].

The dependence on the product sup-norm here is unsatisfactory. For instance, such a product doesn't preserve orthogonal representations. The second observation is the starting point for my following result, which allows us to compare little disk operads over different norms:

Result 3. For $p, q \geq 2$, there is a weak equivalence of equivariant operads

$$\mathscr{D}_{V \overset{p}{\oplus} W} \simeq \mathscr{D}_{V \overset{q}{\oplus} W}$$

where $V \overset{p}{\oplus} W$ is $V \oplus W$ with the p-product norm.

[1] For a product p-norm I mean $||(v,w)|| := ||(||v||, ||w||)||_p$. This forms a norm on $V \oplus W$ where G still acts isometrically.

The idea is reminiscent of the usual proof of the equivalence of finite dimensional topological vector spaces given by p-norms. We can choose higher and higher k's to define embeddings of the operads as symmetric sequences where each alternate operad is weakly equivalent to each other:

$$\mathscr{D}^{k_1}_{V \overset{p}{\oplus} W} \hookrightarrow \mathscr{D}^{k_2}_{V \overset{q}{\oplus} W} \hookrightarrow \mathscr{D}^{k_3}_{V \overset{p}{\oplus} W} \hookrightarrow \mathscr{D}^{k_4}_{V \overset{q}{\oplus} W}.$$

However, the operadic structure is sensitive to the change in shape, so this doesn't immediately give an equivalence of operads. I get around this by using the Boardman-Vogt resolution to interpolate between different operadic structures.

Hence, we find that the exact norm used isn't that relevant, and we obtain a flexible additivity result. This gives us the additivity of little disks of orthogonal representations as a special case. It also gives the classic result as a special case. In a very satisfying way, it also provides a direct geometric proof of the equivalence between the little disks and little cube operads. While known, this result has only been proven in the literature via an indirect zigzag of equivalences.

One immediate application of this result is that it allows us to automatically impose extra structure for $\mathbb{N}_\infty$-operads modelled by little disk operads. For instance, given some G-universe U, since $U = U \oplus U$ we get that $\mathscr{D}_U = \mathscr{D}_U \otimes \mathscr{D}_U$. Hence, by the defining property of the tensor, any $\mathscr{D}_U$-algebra X is also a $\mathscr{D}_U$-algebra *in $\mathscr{D}_U$-algebras*.

2.2 Norm maps for coinduced Real Bordism

One interesting property of $\mathbb{N}_\infty$-operads is that they are closed under coinduction. This is useful, since given a H-spectrum X which is $\mathscr{O}$-algebra for, the HHR norm $N_H^G X$ is an algebra over the coinduced operad $\mathrm{Coind}_{\mathrm{H}}^{\mathrm{G}}(\mathscr{O}) := \mathrm{Map}^H(G, \mathscr{O})$ which is an $\mathbb{N}_\infty$-operad.

In [Szcb], I study the fixed points of coinduced equivariant little disk operads $\mathrm{Coind}_{\mathrm{H}}^{\mathrm{G}}(\mathscr{D}_V)$. The goal is to understand these better as the fixed points determine what kind of norm maps such an operad encodes, and so gives information on the algebraic structure of the HHR norms $N_H^G X$ for spectra X that are natural candidates for algebras over "$\mathbb{N}_k$-operads". Moreover, perhaps closure under coinduction is a property to expect from $\mathbb{N}_k$-operads, so these are possible models for such operads.

I have shown some general combinatorial results on the homotopy of the fixed points of $\mathrm{Coind}_{\mathrm{H}}^{\mathrm{G}}(\mathscr{D}_V)$. In general, they are determined by the fixed points of the original $\mathscr{D}_V$, which themselves are homotopic to orbit configuration spaces on V by a result of Hill's [Hil22].

Unfortunately, such formulas are not particularly enlightening. However, an interesting case is when we are dealing with the coinduced objects of so-called $\mathbb{E}_\sigma$-algebras, which we can view as $\mathscr{D}_\sigma$-algebras. Here σ is the non-trivial C_2-representation. These $\mathbb{E}_\sigma$-algebras have found use in recent years. For instance, Hahn-Shi [HS20] use $\mathbb{E}_\sigma$-algebras in their proof that the complex orientation for Lubin-Tate spectra $\mathrm{MU} \to E_n$ can be refined to a Real orientation $\mathrm{MU}_{\mathbb{R}} \to E_n$.

In the $\mathbb{E}_\sigma$ case for fixed points of the coinduced operads, my results specialize to the following.

Result 4. For $k < s$, the admissible C_{2^k}-sets of $\mathrm{Coind}_{C_2}^{C_{2^s}}(\mathscr{D}_\sigma)$ are of the form

$$T = a \cdot C_{2^k}/e \amalg \epsilon \cdot C_{2^k}/C_{2^k}$$

where $a \in \mathbb{N}$ and $\epsilon = 0, 1$. In this case we have that

$$\mathrm{coind}_{C_2}^{C_{2^s}}(\mathscr{D}_\sigma)_n^\Gamma \simeq \mathrm{Emb}^{C_2}(2^k a \cdot C_2/e, \sigma)^{2^{s-k}}$$

where Γ is the graph subgroup of T, and $n = |T|$.

In particular, we find that this is nonempty, and so we have that $\mathrm{Coind}_{C_2}^{C_{2^s}}(\mathscr{D}_\sigma)$-algebras X have norm maps of the type

$$\bigvee_a (\vee_{C_{2^s}} X) \bigvee_\epsilon X \to X.$$

The norms of Real Bordism $MU^{((G))} := N_{C_2}^G MU_{\mathbb{R}}$ and related spectra are a significant area of research and feature heavily in modern computations in chromatic homotopy theory. Recent work by Beaudry-Hill-Lawson-Shi-Zeng [Bea+22] has investigated the norms of quotients of real bordism by permutation summands:

$$Q := \mathrm{MU}_{\mathbb{R}}^{((G))}/(G \cdot x_1, \dots, G \cdot x_n).$$

We can apply our results to this case as follows: Due to the additivity, we can view $\mathrm{MU}_{\mathbb{R}}$ as a $\mathbb{E}_\rho$-algebra in C_2 –$\mathbb{E}_\infty$- algebras. We then get the following result by norming this up and using a result from Hahn and Wilson [HW18].

Result 5. For $G = C^{2^n}$, the quotients Q are $\mathrm{Coind}_{C_2}^G(\mathscr{D}_\sigma)$-algebras in $\mathrm{MU}_{\mathbb{R}}^{((G))}$-modules.

Using our previous result, we uncover that these quotients have norm maps of the type described above. This may prove useful for future calculations.

3 Future Work

3.1 $\mathbb{N}_k$-operads

While equivariant little disks are good candidates for $\mathbb{N}_k$-operads, we don't expect them to model all possible such operads. In the $\mathbb{N}_\infty$-case, Blumberg and Hill show that there are linear isometry operads that are equivalent to any little disk operads. Since the $\mathbb{N}_k$-operads should describe the same range of possible norm maps, just encoding less higher homotopic information, we don't expect that the little disks are sufficient. This behaviour is in contrast to the non-equivariant case, where such operads can be defined as weakly equivalent to the little disks.

Instead, we could construct these operads more abstractly. One idea is to use Barwick's operator categories [Bar18]. The category of finite ordered sets **O** is the category that controls the range of possible multiplications of non-symmetric operads. Barwick's theory shows how to construct both (non-symmetric) Complete Segal Operads and Lurie's ∞-operads. This gives us a weak notion of operad instead of the strict kind I've used up to this point. Here, the $\mathbb{E}_1$-operads are given by cofibrant replacement of the unit operads. The higher $\mathbb{E}_k$-operads are given as cofibrant replacement of the unit operads over the wreath iterates $\mathbf{O}^{(n)} := \mathbf{O} \wr \mathbf{O} \wr \cdots \wr \mathbf{O}$. Since we know combinatorially what multiplications to expect, it seems possible to use (some extension) Barwick's theory to construct a notion of $\mathbb{N}_k$-(∞-)operads.

Question 2. Can we construct a model for (the homotopy category of) $\mathbb{N}_k$-operads by using Barwick's operator categories and picking the correct operator categories?

An abstract definition like the above would be difficult to work with, so more geometric models would be preferable. In the $\mathbb{N}_\infty$, there are equivalent little disk operads over non-equivalent representations. For instance, if I, I' are non-isomorphic nontrivial representations of C_5, then we have that $\mathscr{D}_{(\mathbb{R}\oplus I)^\infty} \simeq \mathscr{D}_{(\mathbb{R}\oplus I')^\infty}$. I suspect something similar happens in the finite case. Understanding this may provide clues into how we can change the little disk operads and construct better geometric models for $\mathbb{N}_k$-operads.

Question 3. Do we similarly get equivalences in the finite case? i.e., is there a weak equivalence

$$\mathscr{D}_{\mathbb{R}\oplus I} \simeq \mathscr{D}_{\mathbb{R}\oplus I'}?$$

One possible way to approach this might be to extend Berger's theory of cellular $\mathbb{E}_k$-operads [Ber]. Berger defines cellular $\mathbb{E}_n$-operads as sub-operads of combinatorial $\mathbb{E}_\infty$-operads. The idea would be to extend this equivariantly and then use the $\mathbb{N}_\infty$ case.

3.2 Equivariant Additivity

While I have proven additivity for the little disks, the question arises: what other operads have an additivity property? In some respects, additivity is a defining property for $\mathbb{E}_k$-operads, and so any notion of $\mathbb{N}_k$-operads should also have such a property. In particular, non-equivariantly, it has long been held that cofibrant $\mathbb{E}_k$ operads have an additivity property [FV15][2]

An essential part of my proof of additivity for little disks is the notion of "separateness," which allows us to restrict the size of little disks. Intuitively, this implies that the homotopy of little disks depends only on the relative configurations of the centres of the disks. However, this idea is already captured in the non-equivariant case by the Fulton-Macpherson Operad $\mathscr{F}_m$, which is weakly equivalent to little disks. This leads me to conjecture the following

Conjecture 1. There is a weak equivalence of operads

$$\mathscr{F}_m \otimes \mathscr{F}_n \simeq \mathscr{F}_{m+n}$$

[2] It isn't well known, but this paper has a mistake in it, so the status of this result is unclear. Either way, it is almost universally expected that such a result should hold in some form.

which can also be extended equivariantly.

Moreover, $\mathscr{F}_m$ are almost cofibrant. So this would be an interesting connection between the two approaches to additivity in the classic strict models – the combinatorial one via universal cofibrant models and the geometric one by little disks.

Equivariance can also feature more strongly in additivity. In particular, since the Boardman-Vogt Tensor is symmetric monoidal, we can define a corresponding tensor induction

$$\mathrm{ind}_H^G : \mathrm{Op}(H) \to \mathrm{Op}(G).$$

This allows us to construct G-operads, and understanding the homotopical properties of this functor would be interesting. In connection with additivity, we would expect the following holds

Conjecture 2. For a H-representation V, we have a weak equivalence of G-operads

$$\mathrm{ind}_H^G(\mathscr{D}_V) \simeq \mathscr{D}_{\mathrm{ind}_H^G V}.$$

More generally, for a good notion of $\mathbb{N}_V$-operads there is a notion of induction

$$\mathrm{ind}_H^G : \mathbb{N}_V\text{-operads} \to \mathbb{N}_{\mathrm{ind}_H^G V}\text{-operads}$$

with good homotopical properties.

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